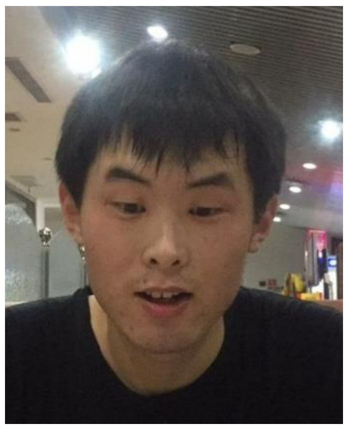
Tight Dynamic Problem Lower Bounds from Generalized BMM and OMv



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STOC 2022

Dynamic problems

- Maintain some data D (graphs, sequences, …)
- Support small updates to D (insertions, deletions, …)
- Answer queries about D (connectivity?...)

- Best possible update time and query time?

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- **Higher LBs from <u>Fine-Grained Conjectures</u>!**
  - A long line of work
    - [Pătraşcu STOC'10]
    - [Abboud and Vassilevska Williams FOCS'14]
    - [Henzinger, Krinninger, Nanongkai, and Saranurak STOC'15]

    …

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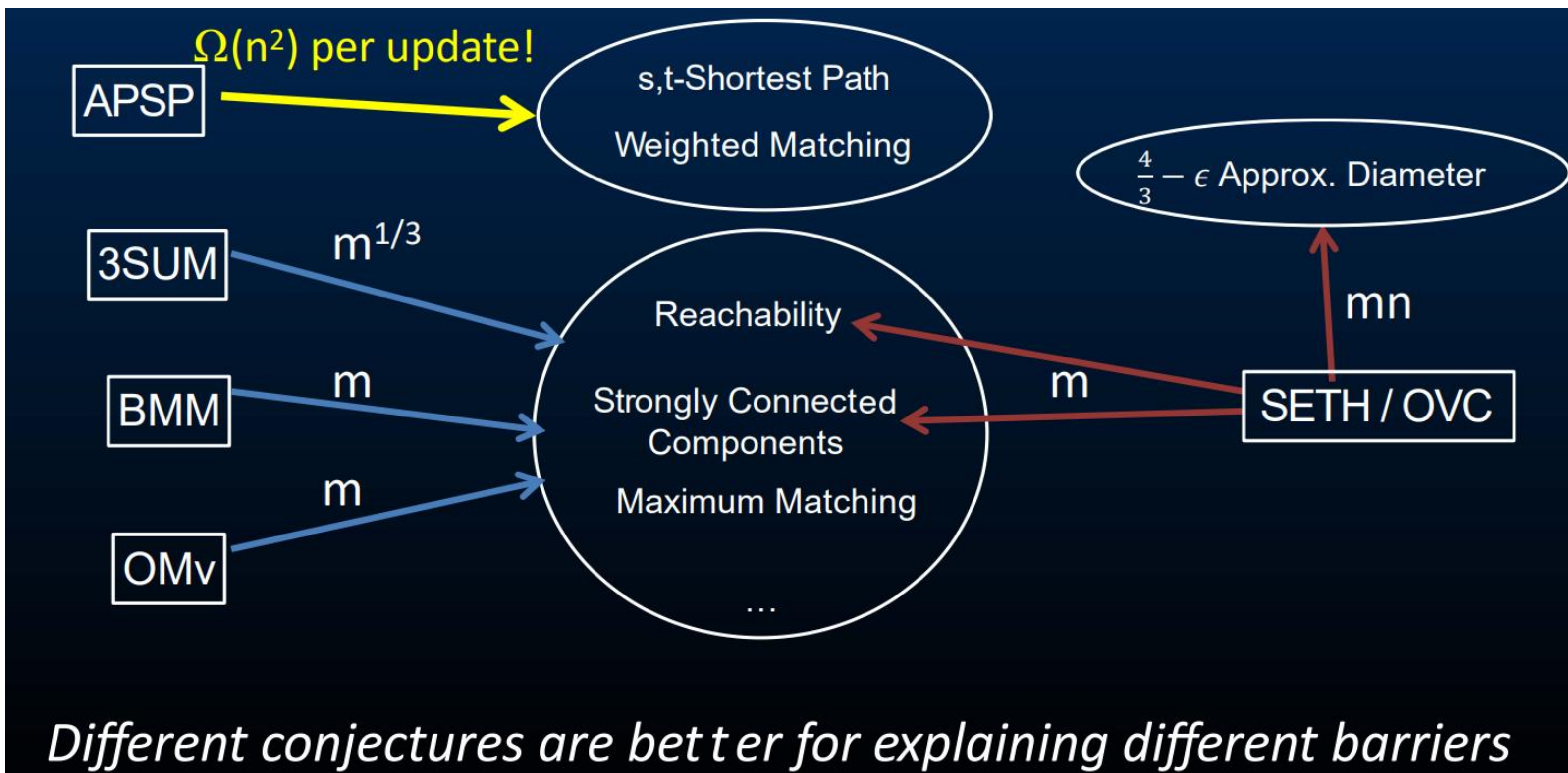

(from Virginia's lecture notes)

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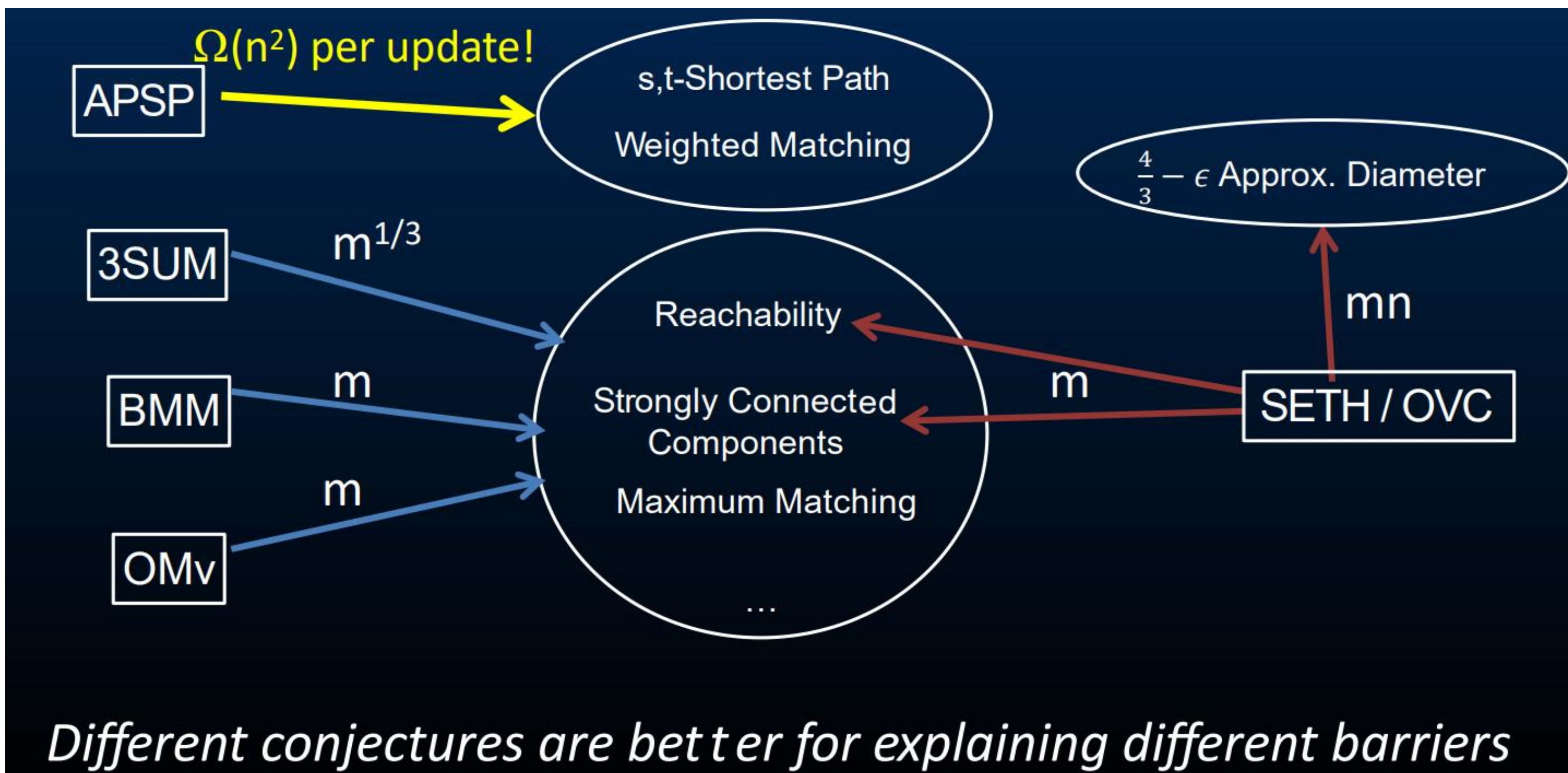

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This work:
A collection of new Lower Bounds based on generalized versions of **BMM** and **OMv**

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- Input: $n \times n$ Boolean matrices A B
- Output: Boolean matrix product AB

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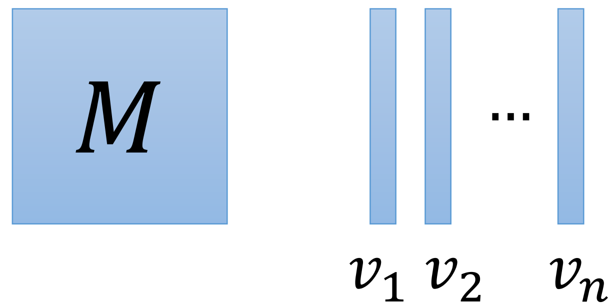
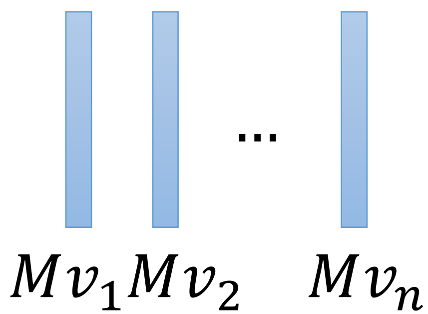
  - Current best: $\boldsymbol{n^3}(\log\log n)^{O(1)}/(\boldsymbol{\log n})^{\boldsymbol{4}}$ [Yu'15]

Algorithms avoiding Fast Matrix Multiplication (e.g. Strassen's)

OMv hypothesis [Henzinger-Krinninger-Nanongkai-Saranurak STOC'15]

- Input: Boolean 

- Output: 

(in an online fashion)

OMv hypothesis [Henzinger-Krinninger-Nanongkai-Saranurak STOC'15]

- Input: Boolean M $v_1\ v_2\ \ldots\ v_n$
- Output: $Mv_1\ Mv_2\ \ldots\ Mv_n$ (in an online fashion)
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Not only "combinatorial"

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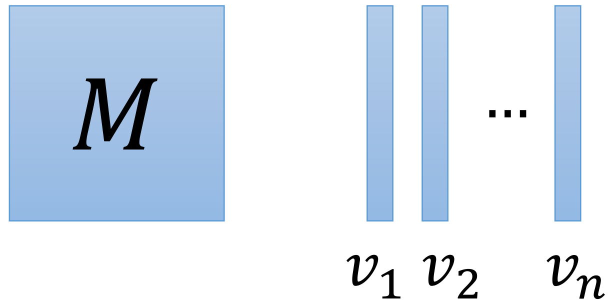

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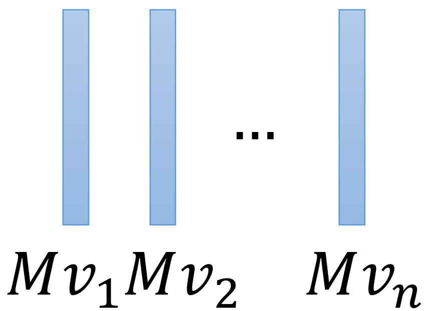

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- Conjecture: No $O(n^{3-\varepsilon})$-time algorithm exists
  - Current best: $n^3/2^{\Omega\left(\sqrt{\log n}\right)}$ time [Larsen-Williams'17]

Dynamic Range-Mode Query

- Maintain an integer array $A[1], A[2], \ldots, A[n]$
- Support Insertions and Deletions
- Query l, r: what is the most frequent element in $A[l], A[l+1], \ldots, A[r]$?
  (breaking ties arbitrarily)

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- Can these combinatorial algorithms be improved?

Dynamic Range-Mode Query: Lower Bounds

- *Static* Range-Mode: Tight combinatorial LB
  - Under BMM, no combinatorial algorithm can achieve $\boldsymbol{n^{0.5-\varepsilon}}$ query time and $n^{1.5-\varepsilon}$ preprocessing time [Chan-Durocher-Larsen-Morrison-Wilkinson’14]

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- *Dynamic* Range-Mode LB (NOT Tight ☹):
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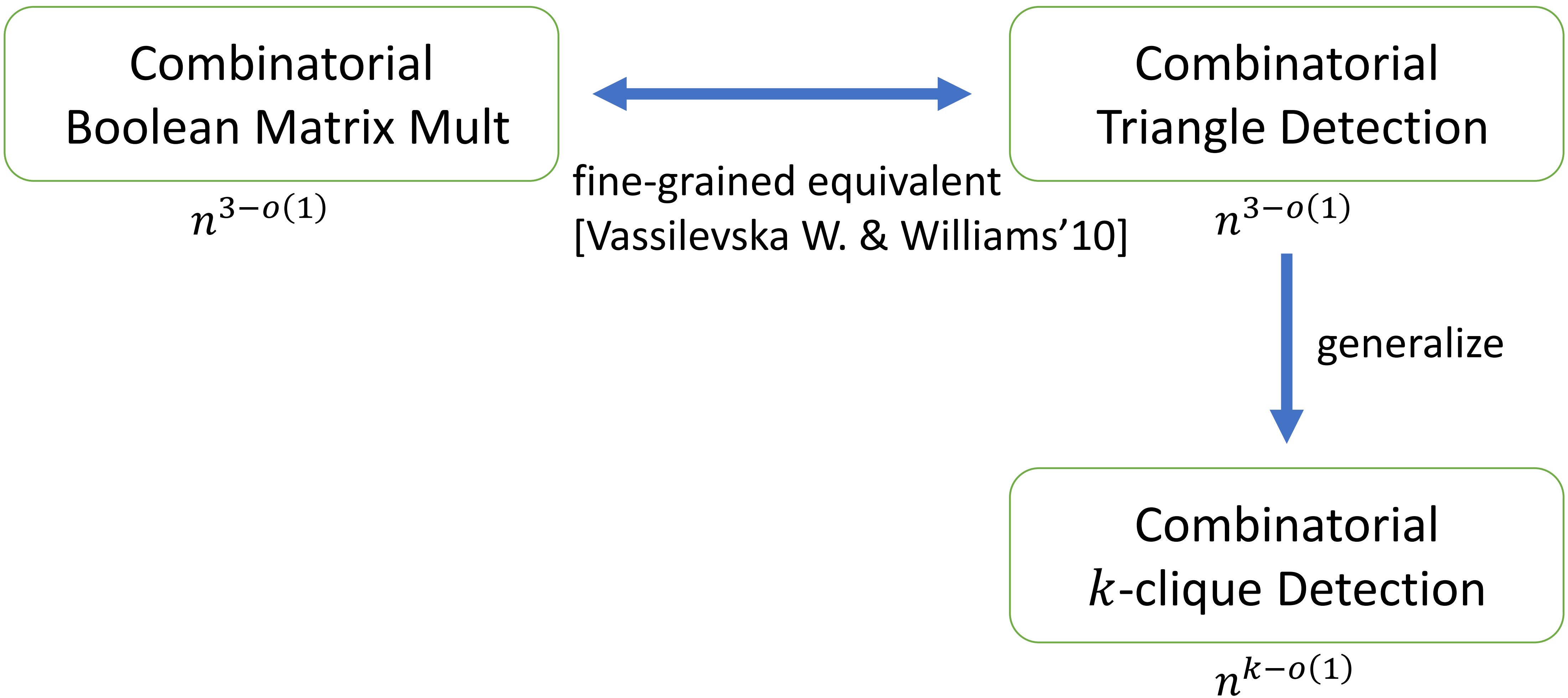
Combinatorial k-clique hypothesis: No combinatorial algorithm can detect k-clique in an n-node graph in $O\left(n^{k-\varepsilon}\right)$ time



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A proof template

Static LB from
$(k-1)$-clique hypothesis

Use dynamic operations to efficiently enumerate the extra k-th node

Dynamic LB from
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Chan et al.'14: *Static* Range Mode requires $n^{0.5-o(1)}$ time (from combinatorial 3-clique)

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Tight combinatorial LBs for more dynamic problems:

- Dynamic 2D Orthogonal Range Color Counting $n^{2/3-o(1)}$ time ($k = 4$)
- Dynamic d-Dimensional Orthogonal Range Mode $n^{1-\frac{1}{2d+1}-o(1)}$ time ($k = 2d + 2$)
- Dynamic 2-Pattern Document Retrieval $n^{2/3-o(1)}$ time ($k = 4$)
- ...

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Main takeaway:
(Combinatorial) k-clique hypothesis is useful for dynamic lower bounds!

Previous dynamic LBs mostly used $k = 3$ (BMM).
(exception: [Gutenberg, Vassilevska Williams, and Wein'20] reduction from 4-clique to dynamic shortest path)

Dynamic Subgraph Connectivity

- Preprocess a static undirected graph G with m edges
- Maintain a dynamic <u>vertex subset S</u> ("on" vertices)
  - Turn on u: $S \leftarrow S \cup \{u\}$
  - Turn off u: $S \leftarrow S \setminus \{u\}$
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- Combinatorial algorithm by Chan, Pătraşcu, and Roditty (FOCS'08) in
  - $\boldsymbol{\tilde{O}(m^{2/3})}$ update time (amortized)
  - $\tilde{O}(m^{1/3})$ query time
  - ($\tilde{O}(m^{4/3})$ preprocessing time)
- Can the $2/3$ exponent be improved?

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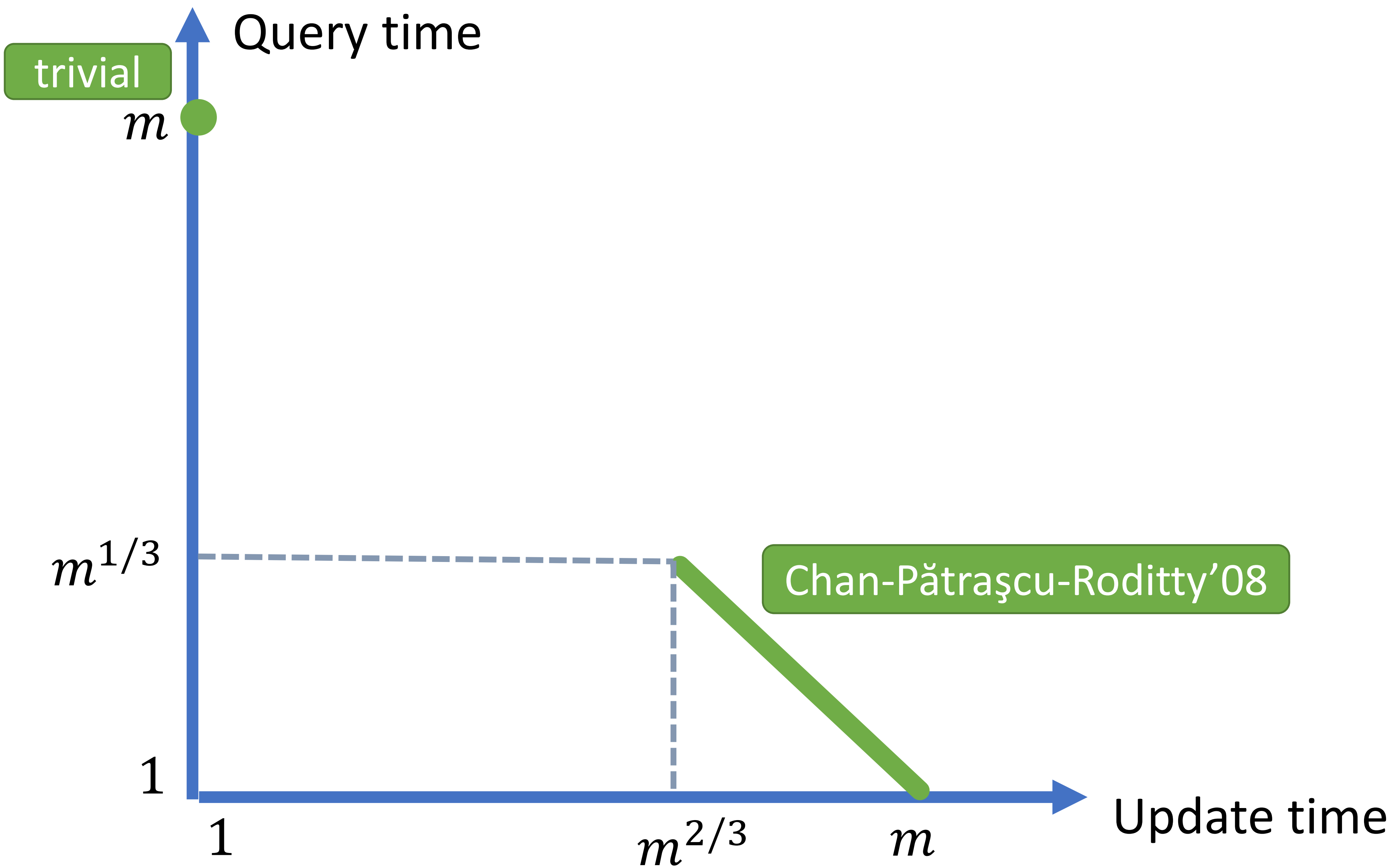

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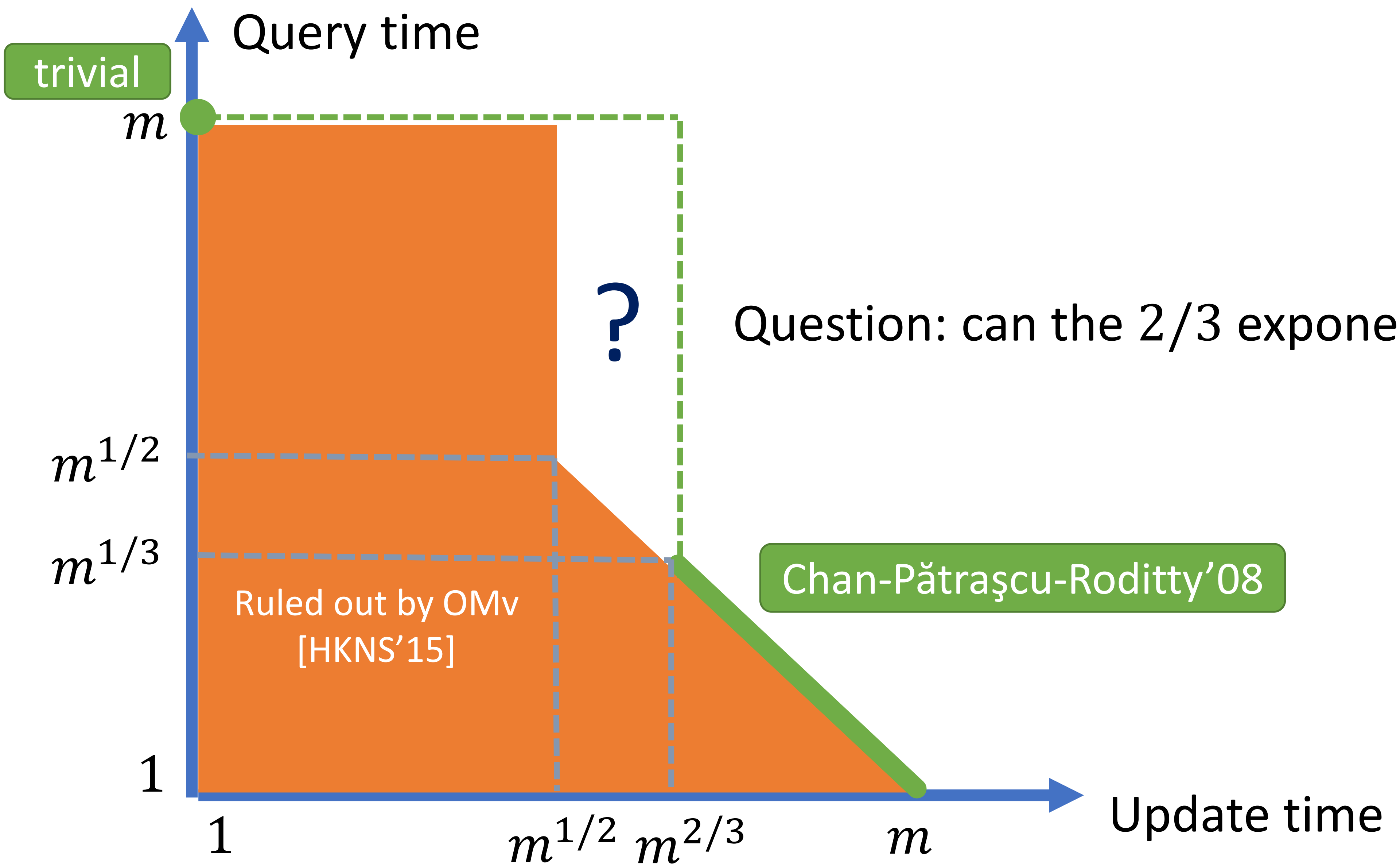

Question: can the 2/3 exponent be improved?

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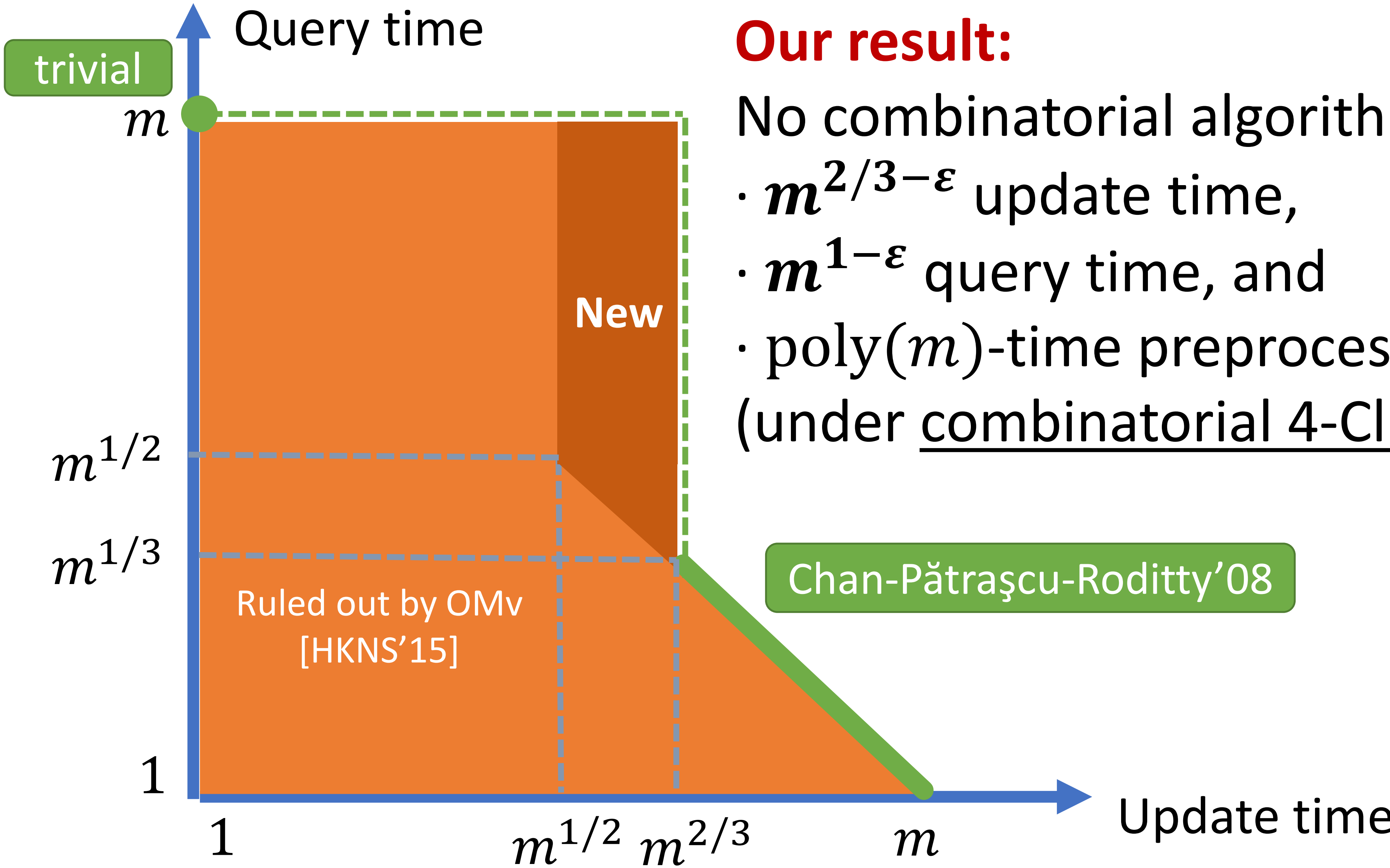


Our result:

No combinatorial algorithm can achieve

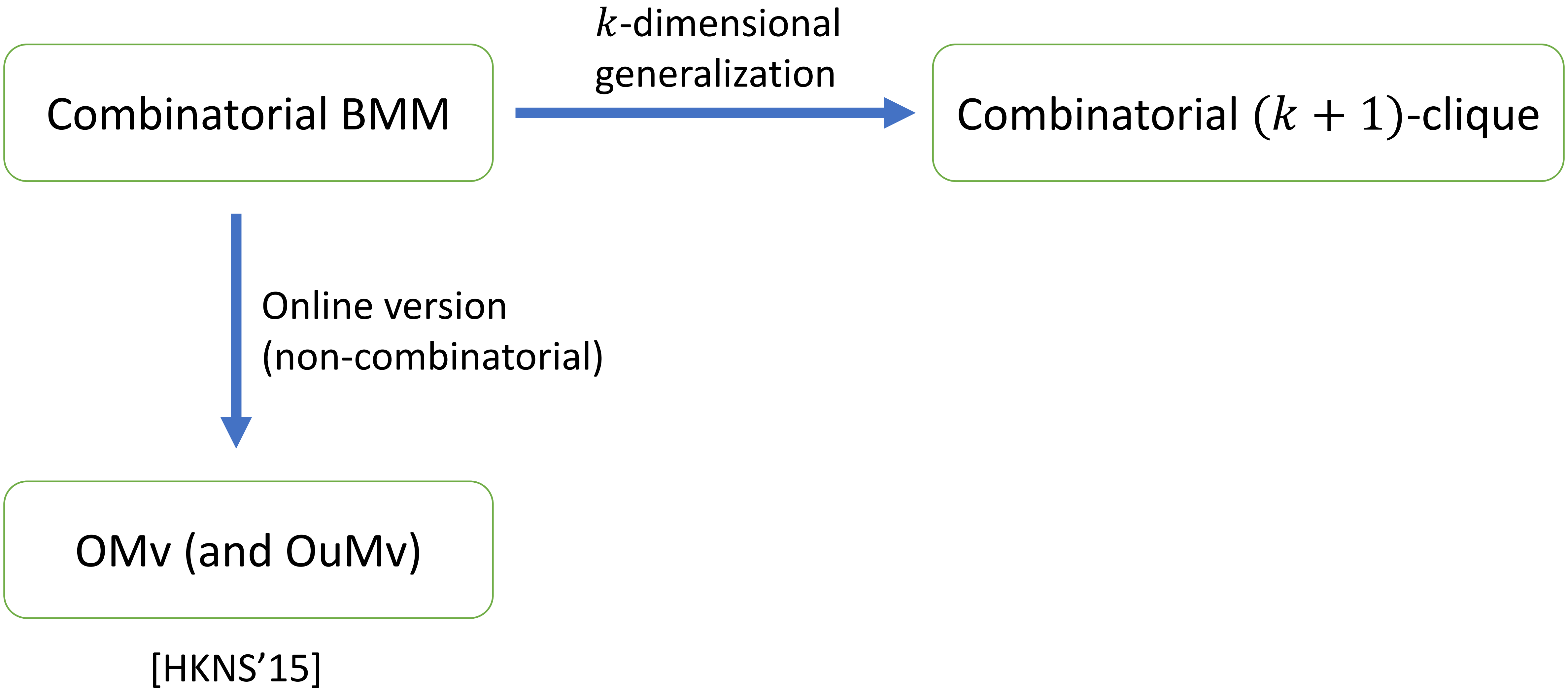
· $\boldsymbol{m^{2/3-\varepsilon}}$ update time,

· $\boldsymbol{m^{1-\varepsilon}}$ query time, and

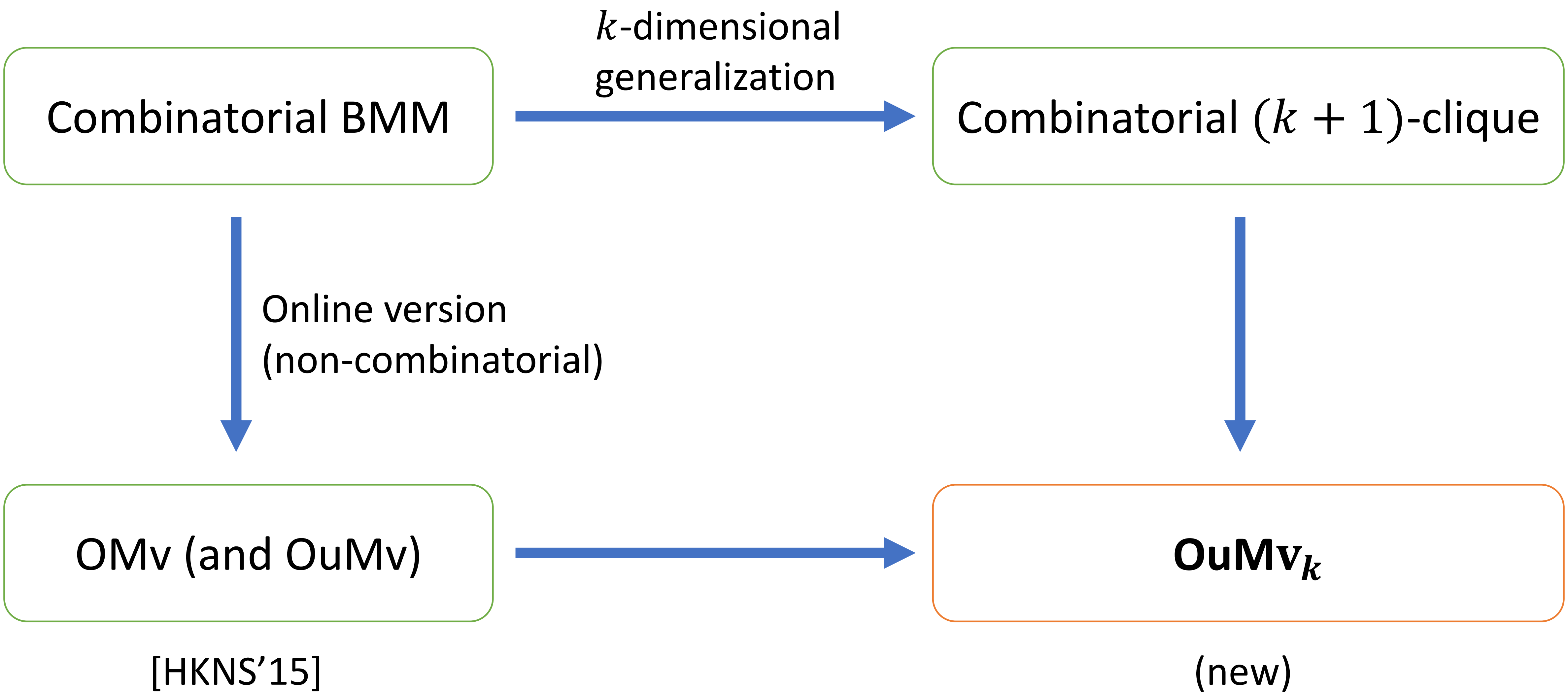
· $\mathrm{poly}(m)$-time preprocessing time

(under <u>combinatorial 4-Clique hypothesis</u>)

A new fine-grained conjecture



A new fine-grained conjecture



OuMv_k hypothesis

- Pre-process a subset $M \subseteq \{1,2,\ldots,n\}^k$
- Answer n online queries:
  - Given k sets $U^{(1)}, U^{(2)}, \ldots, U^{(k)} \subseteq \{1,2,\ldots,n\}$,
  - Is $\left(U^{(1)} \times U^{(2)} \times \cdots \times U^{(k)}\right) \cap M$ non-empty?
- Conjecture: No $O\left(n^{1+k-\varepsilon}\right)$-time algorithm exists

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- Naturally generalizes OuMv [HKNS’15] (which is OuMv_2)

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- Naturally generalizes OuMv [HKNS'15] (which is OuMv_2)
- Useful for dynamic geometry problems in $\mathbf{R}^k$
  - Obtain higher lower bounds as dimension k increases

Dynamic Skyline (Maximal) Points Counting

- Maintain a set P of n points in $\mathbf{R}^d$
- Insertion: $P \leftarrow P \cup \{x\}$
- Deletion: $P \leftarrow P \setminus \{x\}$
- Query: how many "skyline points" does P have?
  - $x \in P$ is a "skyline point"("maximal point") iff no other $y \in P$ dominates x (i.e. $y_i \geq x_i$ for all $i = 1,2, \dots, d$)

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Chan'03 (adapted): A *semi-online* algorithm in $\mathbf{R}^{2k-1}$ with $\tilde{O}\left(n^{1-1/k}\right)$ update time.

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Our result: this is <u>tight</u> under **OuMv$_k$ hypothesis**

(The $k = 2$ case based on OMv was recently independently proved by Dallant & Iacono (2021))

Conclusion

- We used combinatorial k-clique hypothesis and OuMv_k hypothesis to prove tight fine-grained lower bounds for dynamic problems.

Open questions:

- Can Dynamic Subgraph Connectivity have update time better than $m^{2/3}$ using fast matrix multiplication?
- What is the optimal update time for Dynamic Skyline Points Counting in $\mathbf{R}^{2k}$? (semi-online algorithms allowed)

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