Simulating Random Walks on Graphs in the Streaming Model

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Problem Definition

Insertion-only graph streaming model

Let G be the (directed or undirected) input graph with n vertices.
The edges of G come as an input stream $(e_1, e_2, \ldots, e_m)$.
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A sequence of vertices $(v_0, v_1, \ldots, v_t)$ starting from v_0.
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Our problem: Simulating a t-step random walk

A starting vertex v_0 is given at the end of the input stream.
The streaming algorithm outputs a random **sequence** $(v_0, v_1, \ldots, v_t)$.
The ℓ_1 distance between the output distribution and the distribution of t-step random walks is less than ε.

A simple algorithm

Reservoir Sampling

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Main questions

Can we do better (when small error $\varepsilon > 0$ is allowed)?

Can we prove space lower bounds?

Related work

In the multi-pass streaming model: Algorithm using $O(n)$ space and $O(\sqrt{t})$ passes. [Das Sarma, Gollapudi, Panigrahy, 2011]

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Our study: What can we do in the single-pass streaming model?

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Space lower bound for undirected graphs

We show a reduction from the INDEX problem.

INDEX problem

Alice has an n-bit vector $X \in \{0,1\}^n$ and Bob has an index $i \in [n]$. Alice sends a message to Bob, and then Bob should output the bit X_i.

INDEX lower bound [Miltersen, Nisan, Safra, Wigderson, 1998]

For any constant $1/2 < c \leq 1$, solving the INDEX problem with success probability c requires sending $\Omega(n)$ bits.

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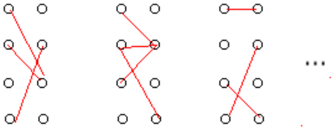

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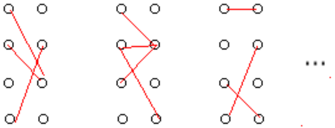

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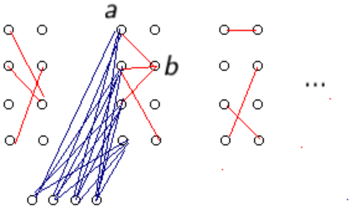

Bob wants to see whether edge (a, b) exists.
Alice sends the memory of the streaming algorithm to Bob. Bob adds $\sqrt{t}$ vertices and connect each of them to every vertex in a's side.

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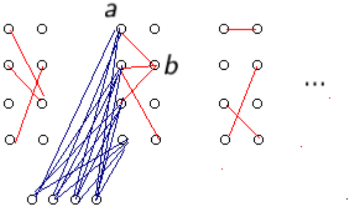

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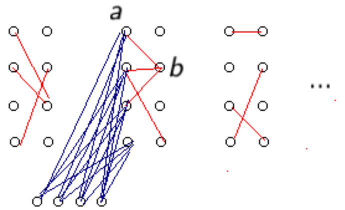

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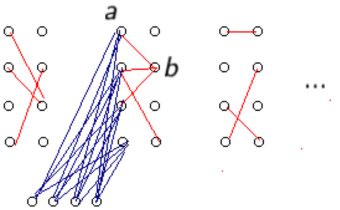

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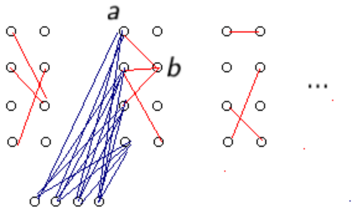

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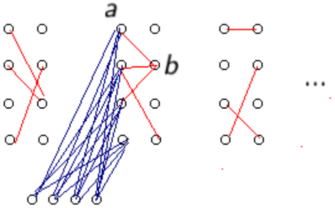

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By INDEX lower bound, we need $\Omega(n\sqrt{t})$ bits of space.

Algorithm for undirected graphs

(For now we assume there are no multiple edges or self-loops)

- Small vertices: degree $\leq C$
- Big vertices: degree $> C$

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Total space: $O(n\sqrt{t})$ words.

Algorithm for undirected graphs using $O(n\sqrt{t})$ space

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If we can make Pr[FAIL]$\leq \varepsilon$, then our output distribution will be (2ε)-close.

Analysis of failure probability

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$$\text{where } \Pr[(v_0, v_1, \ldots, v_t)] = \frac{1}{d(v_0)d(v_1)\ldots d(v_{t-1})}$$

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"$u \to$ Big" steps becomes "Big $\to u$" steps!

Algorithm for undirected graphs

Hence

$$\Pr[u \text{ fails}] \le n \sum_{\text{walk } w} \Pr[w] \cdot 1[w \text{ has more than } C \text{ ``Big} \to u\text{'' steps}]$$

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Hence

$$\Pr[u \text{ fails}] \leq n \sum_{\text{walk } w} \Pr[w] \cdot 1[w \text{ has more than } C \text{ ``Big} \to u\text{'' steps}]$$

Big vertex has degree $> C$. $\Pr\left[v_i \to v_{i+1} \text{ is a ``Big} \to u\text{'' step} \mid v_i\right] < 1/C$
In t steps, "Big $\to u$" happens $< t/C$ times in expectation (and it has concentration!)
Choosing $C = O(\sqrt{t} \log n\varepsilon^{-1})$ makes $\Pr[\text{FAIL}] \leq \sum_u \Pr[u \text{ fails}] \ll \varepsilon$.

(we can improve the log-factor using more careful analysis..)

Dealing with multiple edges

When there are multiple edges,

$$\Pr\left[v_i \to v_{i+1} \text{ is a ``Big} \to u\text{'' step} \mid v_i\right] < 1/C$$

might not hold. (example: most of v's adjacent edges are connecting u)

Fix: For each u, we use heavy-hitter algorithms to find those neighbors v such that $\Pr[v_i = v \mid v_{i-1} = u] > 1/C$.

Thank you!