Cooperation via Codes in Restricted Hat Guessing Games

Kai Jin (HKUST)
Ce Jin (Tsinghua University)
Zhaoquan Gu (Guangzhou University)

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- circuit complexity
- network coding
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There are many variations of the Hat Guessing game.

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- A cooperative team of n players, and T hats with *distinct colors* $1, \ldots, T$
- The dealer *uniformly randomly* places k hats to each player, and d hats remain in the dealer's hand. ($T = nk + d$)
- Each player sees the hats of all other players, but cannot see the hats of his (her) own.

Game definition

(n players and T distinct hats. Each player gets k hats. $d = T - nk \geq 1$ hats remain.)

- Each player guesses k colors. The guess is right iff they exactly match the k colors (s)he receives.
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We consider two winning rules:

- *All-right rule*: The team wins iff **all** players are right
- *One-right rule*: The team wins iff **at least one** player is right

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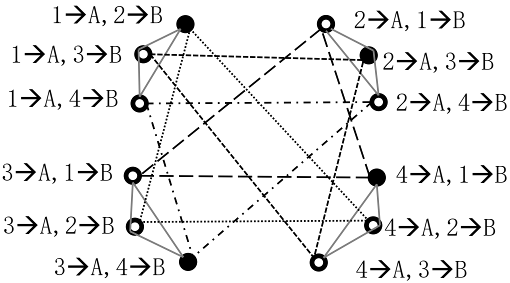
- We determine the exact value of maximum winning probability for some interesting special cases in the all-right rule, and the general case in the one-right rule.
- Constructing explicit best strategies leads to some interesting combinatorial problems. We will study the *Latin matching*, which arises in one of our constructions.

All-right rule: General Case (n, k, d)

Graph $G(n, k, d)$:

- Nodes: all possible placements
- Edge (v_1, v_2): iff there exists a player who cannot distinguish placements v_1 and v_2.



Graph G for $(n, k, d) = (2, 1, 2)$

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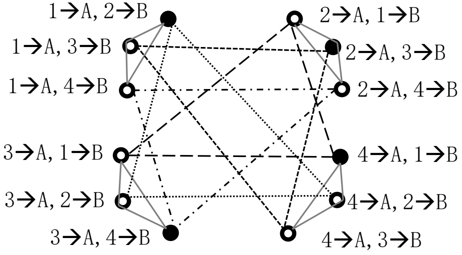

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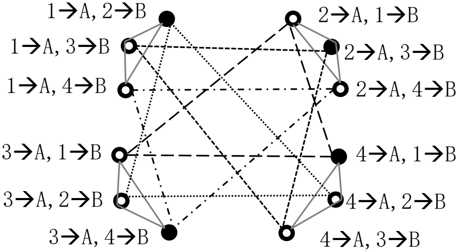

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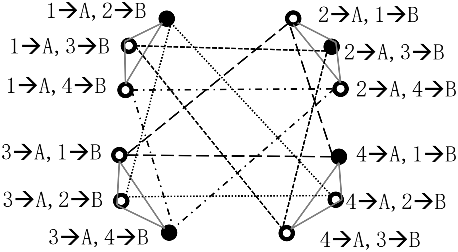

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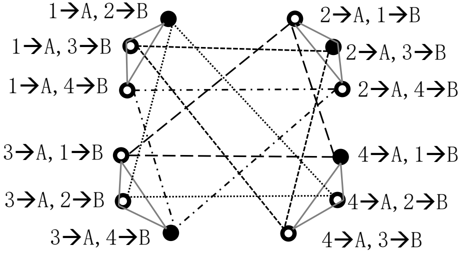

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(In some cases the $1/\binom{k+d}{d}$ upper bound is not achievable. Example: $(n, k, d) = (4, 1, 3)$)

All-right rule: An Interesting Special Case

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Definition: A *Latin matching* f satisfies

- $f : \binom{[2n-1]}{n-1} \to \binom{[2n-1]}{n}$ is a *perfect matching* in the subset lattice, i.e., S must be a subset of $f(S)$. And let $f^+(S)$ denote the only element in $f(S) - S$.
- If S and T differ by exactly one element (i.e., $S = \{x_1, x_2, \ldots, x_{n-2}, y\}$, $T = \{x_1, x_2, \ldots, x_{n-2}, z\}$), then $f^+(S) \neq f^+(T)$.

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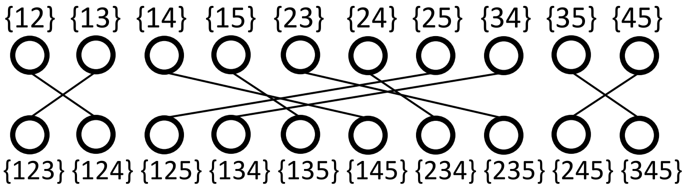
Example of Latin matchings:

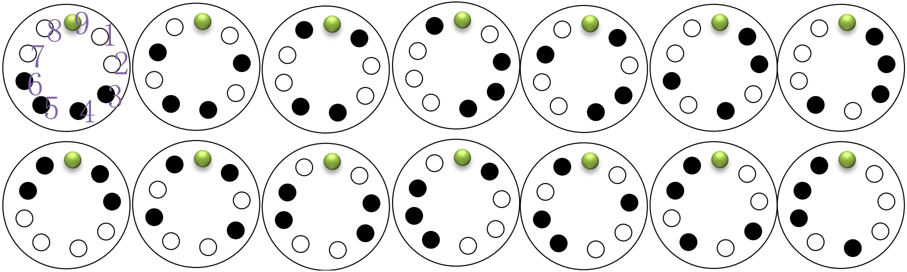
- $n = 2$: $f(\{1\}) = \{1, 2\}, f(\{2\}) = \{2, 3\}, f(\{3\}) = \{3, 1\}$.
- $n = 3$:

$(f')^+$	1	2	3	4	5
1	–	4	2	5	3
2	4	–	5	3	1
3	2	5	–	1	4
4	5	3	1	–	2
5	3	1	4	2	–



Latin Matching

Example of Latin matching for $n = 5$:



(Explanation: f is cyclic. Black balls denote S and green ball denotes $f(S) - S$. $f(\{3,4,5,6\}) = \{3,4,5,6,9\}, f(\{2,3,4,5\}) = \{2,3,4,5,8\}$.)

Connection Between Latin Matching and $(n, 1, n-1)$ Case

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Theorem: If Latin matching f exists for n, then

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Proof Sketch. For a placement $\mathbf{a} = (a_1, \ldots, a_n)$, denote set $S_\mathbf{a} := \{a_1, \ldots, a_n\}$. There exists a unique $i \in [n]$ such that $f(S_\mathbf{a} - a_i) = S_\mathbf{a}$. Assign color i to $\mathbf{a}$.

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This n-coloring induces n different independent sets of G.

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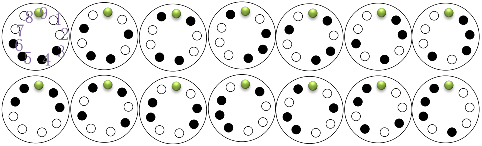
Connection with coding theory:

- The Latin matching construction for $n = 5$ case can be obtained via extended Hamming[8,4,4] codes.

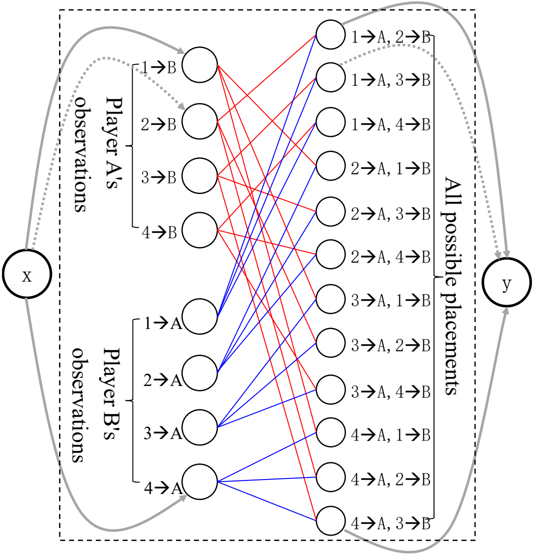


- Application of Latin matchings in our unique-supply variation of Hat Guessing Game is analogous to the application of Hamming codes in the original (red-blue) variation.

One-right Rule

Bipartite graph $H(n, k, d)$:

- Left nodes: possible observations of every player



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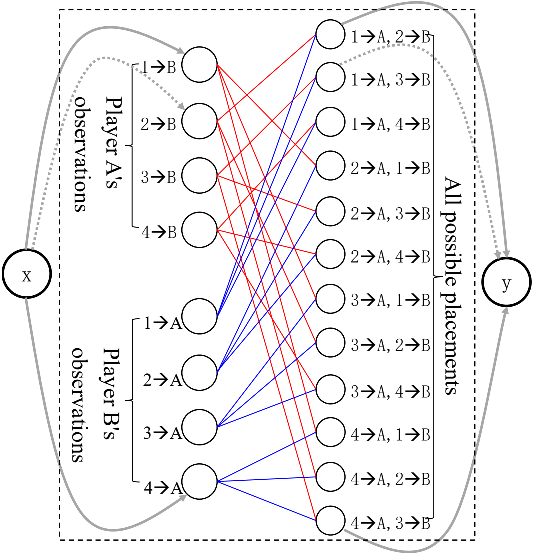

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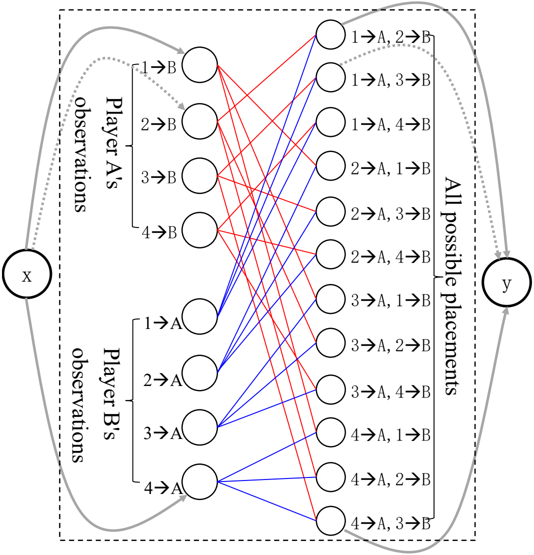
- Edge: observation consistent with placement



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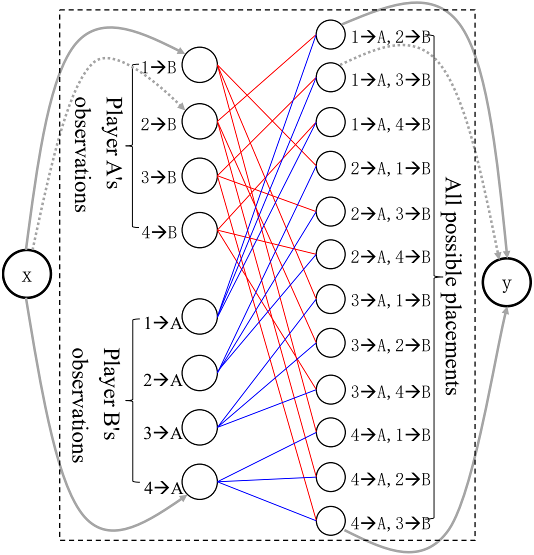
Lemma: The best winning probability in the one-right rule equals $\nu(H)/|G|$, where $\nu(H)$ denotes the **maximum matching** size of graph H.

Theorem: The best winning probability in the one-right rule equals $\min\{1, n/\binom{k+d}{d}\}$.



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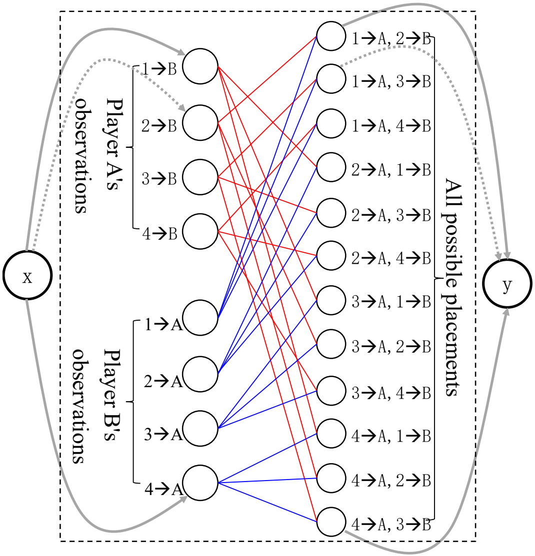
Proof Sketch. H is a regular bipartite graph (vertices on the same side has the samd degree). This implies that H has a complete matching.



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Discussion & Future research

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Can we find other applications of combinatorial tools (e.g., codes, ordered designs, Latin square/Latin matching) in cooperative multi-player games?

Thank you!