Cooperation via Codes in Restricted Hat Guessing Games

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Hat Guessing Games

Hat Guessing Games have been studied extensively in recent years, due to their connections to

- graph entropy
- circuit complexity
- network coding
- auctions
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There are many variations of the Hat Guessing game.
Game definition

We study the *unique-supply* rule (which is a restricted version of the “finite-supply rule” [BHKL09]):
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- A cooperative team of $n$ players, and $T$ hats with *distinct colors* $1, \ldots, T$
- The dealer *uniformly randomly* places $k$ hats to each player, and $d$ hats remain in the dealer’s hand. ($T = nk + d$)
- Each player sees the hats of all other players, but cannot see the hats of his (her) own.
Game definition

($n$ players and $T$ distinct hats. Each player gets $k$ hats. $d = T - nk \geq 1$ hats remain.)

- Each player guesses $k$ colors. The guess is right iff they exactly match the $k$ colors (s)he receives.
- All players guess simultaneously. No communication is allowed after game starts.
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Design a cooperative strategy to maximize winning probability. We consider two winning rules:

- **All-right rule**: The team wins iff all players are right
- **One-right rule**: The team wins iff at least one player is right
Game definition

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A simple observation: The probability that player \( i \) is right is \( 1/\binom{k+d}{d} \).
Game definition

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In All-right rule: winning probability \( \leq \frac{1}{\binom{k+d}{d}} \).

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Game definition

(n players and $T$ distinct hats. Each player gets $k$ hats. $d = T - nk \geq 1$ hats remain.)

A simple observation: The probability that player $i$ is right is $1/(^{k+d}_d)$.

In All-right rule: winning probability $\leq 1/(^{k+d}_d)$.
In One-right rule: winning probability $\geq 1/(^{k+d}_d)$. 

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Our contributions

- We present general methods to compute best strategies in both winning rules.

- We determine the exact value of maximum winning probability for some interesting special cases in the all-right rule, and the general case in the one-right rule.

- Constructing explicit best strategies leads to some interesting combinatorial problems. We will study the Latin matching, which arises in one of our constructions.
Our contributions

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Our contributions

- We present general methods to compute best strategies in both winning rules.
- We determine the exact value of maximum winning probability for some interesting special cases in the all-right rule, and the general case in the one-right rule.
- Constructing explicit best strategies leads to some interesting combinatorial problems. We will study the *Latin matching*, which arises in one of our constructions.
All-right rule: General Case \((n, k, d)\)

Graph \(G(n, k, d)\):

- **Nodes**: all possible placements
- **Edge** \(v_1, v_2\): iff there exists a player who cannot distinguish placements \(v_1\) and \(v_2\).

Graph \(G\) for \((n, k, d) = (2, 1, 2)\):
All-right rule: General Case

(Edge \((v_1, v_2)\) iff there exists a player who cannot distinguish placements \(v_1\) and \(v_2\).)

(Graph \(G\) for \((n, k, d) = (2, 1, 2)\))
All-right rule: General Case

(Edge \((v_1, v_2)\) iff there exists a player who cannot distinguish placements \(v_1\) and \(v_2\).)

\[
\begin{align*}
1 & \rightarrow A, 2 \rightarrow B \\
1 & \rightarrow A, 3 \rightarrow B \\
1 & \rightarrow A, 4 \rightarrow B \\
3 & \rightarrow A, 1 \rightarrow B \\
3 & \rightarrow A, 2 \rightarrow B \\
3 & \rightarrow A, 4 \rightarrow B \\
2 & \rightarrow A, 1 \rightarrow B \\
2 & \rightarrow A, 3 \rightarrow B \\
2 & \rightarrow A, 4 \rightarrow B \\
4 & \rightarrow A, 1 \rightarrow B \\
4 & \rightarrow A, 2 \rightarrow B \\
4 & \rightarrow A, 3 \rightarrow B
\end{align*}
\]

(Graph \(G\) for \((n, k, d) = (2, 1, 2)\))

**Theorem**: The best winning probability in the all-right winning rule equals \(\alpha(G)/|G|\), where \(\alpha(G)\) denotes the maximum independent set size of \(G\).
All-right rule: General Case

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1 &\rightarrow A, 2 \rightarrow B \\
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1 &\rightarrow A, 4 \rightarrow B \\
2 &\rightarrow A, 1 \rightarrow B \\
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2 &\rightarrow A, 4 \rightarrow B \\
3 &\rightarrow A, 1 \rightarrow B \\
3 &\rightarrow A, 2 \rightarrow B \\
3 &\rightarrow A, 4 \rightarrow B \\
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**Theorem:** The best winning probability in the all-right winning rule equals \(\alpha(G)/|G|\), where \(\alpha(G)\) denotes the maximum independent set size of \(G\). Example: \(\alpha(G(2, 1, 2)) = 4\), implying that optimal strategy has \(4/12 = 1/3\) winning probability, matching the \(1/(k+d)\) upper bound.
All-right rule: General Case

(Edge $(v_1, v_2)$ iff there exists a player who cannot distinguish placements $v_1$ and $v_2$.)

1→A, 2→B
1→A, 3→B
1→A, 4→B
2→A, 1→B
2→A, 3→B
2→A, 4→B
3→A, 1→B
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(Graph $G$ for $(n, k, d) = (2, 1, 2)$)

**Theorem:** The best winning probability in the all-right winning rule equals $\alpha(G)/|G|$, where $\alpha(G)$ denotes the maximum independent set size of $G$. Example: $\alpha(G(2, 1, 2)) = 4$, implying that optimal strategy has $4/12 = 1/3$ winning probability, matching the $1/(k+d)$ upper bound.

(In some cases the $1/(k+d)$ upper bound is not achievable. Example: $(n, k, d) = (4, 1, 3)$)
All-right rule: An Interesting Special Case

\((n, k, d) = (n, 1, n - 1)\) under all-right rule.

(total number of hats \(T = 2n - 1\); each of the \(n\) players gets one hat)
(n, k, d) = (n, 1, n − 1) under all-right rule. (total number of hats \( T = 2n − 1 \); each of the \( n \) players gets one hat)

**Definition:** A *Latin matching* \( f \) satisfies

- \( f : \binom{[2n − 1]}{n − 1} \rightarrow \binom{[2n − 1]}{n} \) is a *perfect matching* in the subset lattice, i.e., \( S \) must be a subset of \( f(S) \). And let \( f^+(S) \) denote the only element in \( f(S) − S \).
- If \( S \) and \( T \) differ by exactly one element (i.e., \( S = \{x_1, x_2, \ldots, x_{n−2}, y\}, \ T = \{x_1, x_2, \ldots, x_{n−2}, z\} \)), then \( f^+(S) \neq f^+(T) \).
Latin Matching

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Example of Latin matchings:

- \( n = 2 \): \( f(\{1\}) = \{1, 2\}, f(\{2\}) = \{2, 3\}, f(\{3\}) = \{3, 1\} \).

- \( n = 3 \):

| \((f')^+\) | 1 2 3 4 5 |
|---|---|---|---|---|
| 1 | \(\cdot\) 4 2 5 3 |
| 2 | 4 \(\cdot\) 5 3 1 |
| 3 | 2 5 \(\cdot\) 1 4 |
| 4 | 5 3 1 \(\cdot\) 2 |
| 5 | 3 1 4 2 \(\cdot\) |

\[
\begin{align*}
&\{12\} \{13\} \{14\} \{15\} \{23\} \{24\} \{25\} \{34\} \{35\} \{45\} \\
&\{123\} \{124\} \{125\} \{134\} \{135\} \{145\} \{234\} \{235\} \{245\} \{345\}
\end{align*}
\]
Latin Matching

Example of Latin matching for $n = 5$:

(Explanation: $f$ is cyclic. Black balls denote $S$ and green ball denotes $f(S) - S$. $f(\{3, 4, 5, 6\}) = \{3, 4, 5, 6, 9\}$, $f(\{2, 3, 4, 5\}) = \{2, 3, 4, 5, 8\}$.)
Connection Between Latin Matching and \((n, 1, n - 1)\) Case

- \(f : \binom{[2n-1]}{n-1} \to \binom{[2n-1]}{n}\) is a perfect matching in the subset lattice, i.e., \(S\) must be a subset of \(f(S)\). And let \(f^+(S)\) denote the only element in \(f(S) - S\).
- If \(S\) and \(T\) differ by exactly one element (i.e., \(S = \{x_1, x_2, \ldots, x_{n-2}, y\}\), \(T = \{x_1, x_2, \ldots, x_{n-2}, z\}\)), then \(f^+(S) \neq f^+(T)\).

**Theorem:** If Latin matching \(f\) exists for \(n\), then

- \(G(n, 1, n - 1)\) is \(n\)-colorable.
- the best winning probability in all-right rule equals \(1/n\). (matching the \(1/(\binom{k+d}{d})\) upper bound)
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**Proof Sketch.** For a placement \(a = (a_1, \ldots, a_n)\), denote set \(S_a := \{a_1, \ldots, a_n\}\). There exists a unique \(i \in [n]\) such that \(f(S_a − a_i) = S_a\). Assign color \(i\) to \(a\).
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If two placements \(a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)\) have the same color \(i\), then \(a, b\) must differ at \(\geq 2\) coordinates (and thus not adjacent on \(G\)).
Connection Between Latin Matching and \((n, 1, n−1)\) Case

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This \(n\)-coloring induces \(n\) different independent sets of \(G\).
Discussion on Latin Matching

**Theorem:** If Latin matching exists for $n$, then $n$ is a prime number.
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Connection with coding theory:

- The Latin matching construction for $n = 5$ case can be obtained via extended Hamming[$8,4,4$] codes.
- Application of Latin matchings in our unique-supply variation of Hat Guessing Game is analogous to the application of Hamming codes in the original (red-blue) variation.
Bipartite graph $H(n, k, d)$:
- Left nodes: possible observations of every player
- Player A's observations
- Player B's observations
- All possible placements

Bipartite graph $H$ for $(n, k, d) = (2, 1, 2)$
One-right Rule

Bipartite graph $H(n, k, d)$:
- Left nodes: possible observations of every player
- Right nodes: possible placements

Bipartite graph $H$ for $(n, k, d) = (2, 1, 2)$
One-right Rule

Bipartite graph $H(n, k, d)$:

- Left nodes: possible observations of every player
- Right nodes: possible placements
- Edge: observation consistent with placement

Bipartite graph $H$ for $(n, k, d) = (2, 1, 2)$
One-right Rule

**Lemma:** The best winning probability in the one-right rule equals $\nu(H)/|G|$, where $\nu(H)$ denotes the **maximum matching** size of graph $H$.

**Theorem:** The best winning probability in the one-right rule equals $\min\{1, n/(k+d)\}$. 

Bipartite graph $H$ for $(n, k, d) = (2, 1, 2)$
One-right Rule

**Theorem:** The best winning probability in the one-right rule equals \( \min\{1, \frac{n}{k+d}\} \).

**Proof Sketch.** \( H \) is a regular bipartite graph (vertices on the same side have the same degree). This implies that \( H \) has a complete matching.
The optimal strategy for one-right rule obtained from complete matching is not explicitly represented. For some restricted case, e.g., $n = 2$ or $k = 1$, explicit strategies could be obtained via combinatorial constructions.
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Can we show/disprove the existence of Latin matchings for primes $n > 5$? (It is known that cyclic Latin matching does not exist for $n = 7$.)
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Can we find other applications of combinatorial tools (e.g., codes, ordered designs, Latin square/Latin matching) in cooperative multi-player games?
Thank you!